CS 70 Discrete Mathematics and Probability Theory
Spring 2020

Quiz 2

1. [True or False?] For each of the questions, answer TRUE or FALSE. [No need to justify.]

(a) ☐ Suppose you proved the inductive step for a statement $P(n)$ but then discovered that $P(29)$ is false. Thus, $P(1)$ has to be false.

(b) ☐ Suppose you proved the inductive step for a statement $P(n)$ but then discovered that $P(29)$ is false. Then, we cannot say anything about P(50).

(c) ☐ In a stable matching instance where there is a job at the bottom of each candidate's preference list, the job is paired with its least favorite candidate in every stable pairing.

(d) ☐ In a stable matching instance where there is a job at the top of each candidate's preference list, the job is paired with its favorite candidate in every stable pairing.

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2. [Inequality.] Prove by induction on n that if n is a natural number and $x > 0$, then $(1+x)^n \geq 1+nx$.

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3. [Stable Matching.] Suppose that after running the Propose-and-Reject Algorithm with n jobs and n candidates, the pairing that results includes the pair $(1,A)$. Suppose that after a few days job 1 changes its mind, and decides that it does not like candidate A as much as it thought it did (i.e. it put her the last candidate on its preference list). What is the maximum number of rogue couples that result in the existing pairing from such a change to 1's preference list? Give a one or two sentence justification for why the number of rogue couples can be as large as you claim. Also give a one or two sentence justification for why the remaining couples cannot be rogue couples.

1 Bipartite Graph

A bipartite graph consists of 2 disjoint sets of vertices (say L and R), such that no 2 vertices in the same set have an edge between them. For example, here is a bipartite graph (with $L = \{$green vertices$\}$ and $R = \{$red vertices$\}$), and a non-bipartite graph.

Figure 1: A bipartite graph (left) and a non-bipartite graph (right).

Prove that a graph has no tours of odd length if it is a bipartite (This is equivalent to proving that, a graph G being a bipartite implies that G has no tours of odd length).

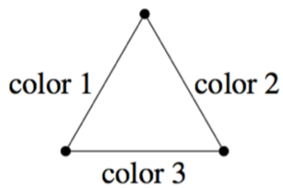
2 Planarity

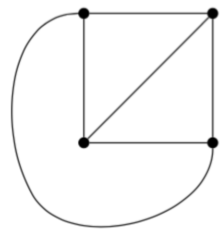
(a) Prove that $K_{3,3}$ is nonplanar.

(b) Consider graphs with the property T: For every three distinct vertices v_1, v_2, v_3 of graph G, there are at least two edges among them. Use a proof by contradiction to show that if G is a graph on ≥ 7 vertices, and G has property T, then G is nonplanar.

3 Edge Colorings

An edge coloring of a graph is an assignment of colors to edges in a graph where any two edges incident to the same vertex have different colors. An example is shown on the left.





(a) Show that the 4 vertex complete graph above can be 3 edge colored. (Use the numbers $1,2,3$ for colors. A figure is shown on the right.)

(b) Prove that any graph with maximum degree $d \geq 1$ can be edge colored with $2d-1$ colors.

(c) Show that a tree can be edge colored with d colors where d is the maximum degree of any vertex.