

1 Count It!

For each of the following collections, determine and briefly explain whether it is finite, countably infinite (like the natural numbers), or uncountably infinite (like the reals):

- (a) The integers which divide 8.
- (b) The integers which 8 divides.
- (c) The functions from \mathbb{N} to \mathbb{N} .
- (d) The set of strings over the English alphabet. (Note that the strings may be arbitrarily long, but each string has finite length. Also the strings need not be real English words.)
- (e) Computer programs that halt. *Hint: How can we represent a computer program?*
- (f) The set of finite-length strings drawn from a countably infinite alphabet, \mathcal{A} .

2 Countability Basics

- (a) Is $f : \mathbb{N} \rightarrow \mathbb{N}$, defined by $f(n) = n^2$, an injection (one-to-one)? Briefly justify.
- (b) Is $f : \mathbb{R} \rightarrow \mathbb{R}$, defined by $f(x) = x^3 + 1$, a surjection (onto)? Briefly justify.

- (c) The Bernstein-Schroder theorem states that, if there exist injective functions $f : A \rightarrow B$ and $g : B \rightarrow A$ between the sets A and B , then a bijection exists between A and B . Use this to demonstrate that $(0, 1)$ and $\mathbb{R}_+ = (0, \infty)$ have the same cardinality by defining appropriate injections.

3 Halting Problem Sanity Check

Suppose you want to prove that a program A is uncomputable. Which of the following should you do?

- (a) Show that A can be solved if the halting problem could be solved.
- (b) Show that the halting problem could be solved if A could be solved.

4 Hello World!

Determine the computability of the following tasks. If it's not computable, write a reduction or self-reference proof. If it is, write the program.

- (a) You want to determine whether a program P on input x prints "Hello World!". Is there a computer program that can perform this task? Justify your answer.

- (b) You want to determine whether a program P prints "Hello World!" before running the k th line in the program. Is there a computer program that can perform this task? Justify your answer.

- (c) You want to determine whether a program P prints "Hello World!" in the first k steps of its execution. Is there a computer program that can perform this task? Justify your answer.