CS 70 Discrete Mathematics and Probability Theory
Spring 2020 Course Notes

DIS 07B

1 Monty Hall Challenge

Let us take on the challenge posed in lecture, and formally analyze the Monty Hall Problem.

(a) Assume that the corgi (the prize) and two goats were placed uniformly at random behind the three doors. What is the probability space $(\Omega, \mathbb{P})$?

(b) If our contestant chose door 1 in the first round, and decides to switch to another door after being shown a goat behind door 2 or 3, what are the events $C_1 =$"They win the corgi" and $\overline{C_1} =$"They win a goat"? What are their probabilities $\mathbb{P}(C_1)$ and $\mathbb{P}(\overline{C_1})$?

(c) If the contestant does not switch doors, what are the events $C_2, \overline{C_2}$ of winning the corgi and goats, and their respective probabilities now?

(d) If instead of choosing door 1 in the beginning, they chose a door uniformly at random, how do your $\Omega, \mathbb{P}, C_i, \overline{C_i}$ from above change?

2 Sample Space and Events

Consider the sample space Ω of all outcomes from flipping a coin 3 times.

(a) List all the outcomes in Ω. How many are there?

(b) Let A be the event that the first flip is a heads. List all the outcomes in A. How many are there?

(c) Let B be the event that the third flip is a heads. List all the outcomes in B. How many are there?

(d) Let C be the event that the first and third flip are heads. List all outcomes in C. How many are there?

(e) Let D be the event that the first or the third flip is heads. List all outcomes in D. How many are there?

(f) Are the events A and B disjoint? Express C in terms of A and B. Express D in terms of A and B.

(g) Suppose now the coin is flipped $n \geq 3$ times instead of 3 flips. Compute $|\Omega|, |A|, |B|, |C|, |D|$.

(h) Your gambling buddy found a website online where he could buy trick coins that are heads or tails on both sides. He puts three coins into a bag: one coin that is heads on both sides, one coin that is tails on both sides, and one that is heads on one side and tails on the other side. You shake the bag, draw out a coin at random, put it on the table without looking at it, then look at the side that is showing. Suppose you notice that the side that is showing is heads. What is the probability that the other side is heads? Show your work. [*Hint*: The answer is NOT $1/2$.]

3 Sampling

Suppose you have balls numbered $1, \ldots, n$, where n is a positive integer ≥ 2, inside a coffee mug. You pick a ball uniformly at random, look at the number on the ball, replace the ball back into the coffee mug, and pick another ball uniformly at random.

(a) What is the probability that the first ball is 1 and the second ball is 2?

(b) What is the probability that the second ball's number is strictly less than the first ball's number?

(c) What is the probability that the second ball's number is exactly one greater than the first ball's number?

(d) Now, assume that after you looked at the first ball, you did *not* replace the ball in the coffee mug (instead, you threw the ball away), and then you drew a second ball as before. Now, what are the answers to the previous parts?