

## 1 Numbered Balls

Suppose you have a bag containing seven balls numbered 0, 1, 1, 2, 3, 5, 8.

- (a) You perform the following experiment: pull out a single ball and record its number. What is the expected value of the number that you record?
  
- (b) You repeat the experiment from part (a), except this time you pull out two balls together and record their total. What is the expected value of the total that you record?

## 2 Airport Revisited

- (a) Suppose that there are  $n$  airports arranged in a circle. A plane departs from each airport, with probability  $1/2$  flies to the airport directly to its left, and with probability  $1/2$  to the one directly to its right. What is the expected number of empty airports after all planes have landed?
  
- (b) Now suppose that we still have  $n$  airports, but instead of being arranged in a circle, they form graph, where each airport is denoted by a vertex, and an edge between two airports indicates that a flight is permitted between them. There is a plane departing from each airport and randomly chooses a neighboring destination where a flight is permitted. What is the expected number of empty airports after all planes have landed? (Express your answer in terms of  $N(i)$ , the set of neighboring airports of airport  $i$ , and  $\deg(i)$ , the number of neighboring airports of airport  $i$ ).

### 3 Ball in Bins

You are throwing  $k$  balls into  $n$  bins. Let  $X_i$  be the number of balls thrown into bin  $i$ .

- (a) What is  $\mathbb{E}[X_i]$ ?
  
- (b) What is the expected number of empty bins?
  
- (c) Define a collision to occur when two balls land in the same bin (if there are  $n$  balls in a bin, count that as  $n - 1$  collisions). What is the expected number of collisions?

### 4 Swaps and Cycles

We'll say that a permutation  $\pi = (\pi(1), \dots, \pi(n))$  contains a *swap* if there exist  $i, j \in \{1, \dots, n\}$  so that  $\pi(i) = j$  and  $\pi(j) = i$ .

- (a) What is the expected number of swaps in a random permutation?
  
- (b) In the same spirit as above, we'll say that  $\pi$  contains a *s-cycle* if there exist  $i_1, \dots, i_s \in \{1, \dots, n\}$  with  $\pi(i_1) = i_2, \pi(i_2) = i_3, \dots, \pi(i_s) = i_1$ . Compute the expectation of the number of *s-cycles*.