CS 70 Spring 2020 — Discrete Mathematics and Probability Theory — HW 1

Due: Friday, January 31, 2020 at 11:59 PM
Grace period until Sunday, February 2, 2020 at 11:59 PM

1 Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student! We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

1. **What sources (if any) did you use as you worked through the homework?**
2. **If you worked with someone on this homework, who did you work with?** List names and student ID's. (In case of homework party, you can also just describe the group.)
3. **How did you work on this homework?** (For example, *I first worked by myself for 2 hours, but got stuck on problem 3, so I went to office hours. Then I went to homework party for a few hours, where I finished the homework.*)
4. **Roughly how many total hours did you work on this homework?**

2 Propositional Logic Language

For each of the following sentences, use the notation introduced in class to convert the sentence into propositional logic. Then write the statement's negation in propositional logic without the use of the $\neg$ symbol.

(a) The cube of a negative integer is negative.

(b) There are no integer solutions to the equation $x^2 - y^2 = 10$.

(c) There is one and only one real solution to the equation $x^3 + x + 1 = 0$.

(d) For any two distinct real numbers, we can find a rational number in between them.

3 Truth Tables

Determine whether the following equivalences hold, by writing out truth tables. Clearly state whether or not each pair is equivalent.

(a) $P \wedge (Q \vee P) \equiv P \wedge Q$

(b) $(P \Rightarrow Q) \Rightarrow R \equiv P \Rightarrow (Q \Rightarrow R)$

(c) $(P \Rightarrow Q) \Rightarrow (P \Rightarrow R) \equiv P \Rightarrow (Q \Rightarrow R)$

(d) $(P \wedge \neg Q) \Leftrightarrow (\neg P \vee Q) \equiv (Q \wedge \neg P) \Leftrightarrow (\neg Q \vee P)$

4 Implication

Which of the following implications are always true, regardless of P? Give a counterexample for each false assertion (i.e. come up with a statement $P(x,y)$ that would make the implication false).

(a) $\forall x \forall y P(x,y) \implies \forall y \forall x P(x,y)$.

(b) $\exists x \exists y P(x,y) \implies \exists y \exists x P(x,y)$.

(c) $\forall x \exists y P(x,y) \implies \exists y \forall x P(x,y)$.

(d) $\exists x \forall y P(x,y) \implies \forall y \exists x P(x,y)$.

5 Proof Practice

(a) Prove that $\forall n \in \mathbb{N}$, if n is odd, then n^2+1 is even.

(b) Prove that $\forall x, y \in \mathbb{R}$, $\min(x,y) = (x+y-|x-y|)/2$.

(c) Prove that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

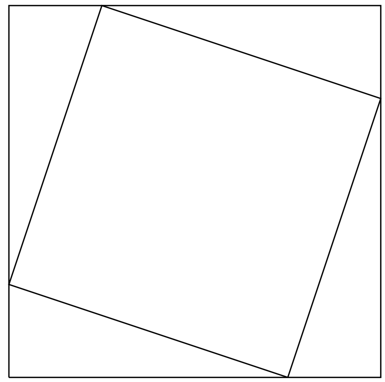
(d) Suppose $A \subseteq B$. Prove $\mathscr{P}(A) \subseteq \mathscr{P}(B)$.

6 Twin Primes

(a) Let $p > 3$ be a prime. Prove that p is of the form $3k+1$ or $3k-1$ for some integer k.

(b) *Twin primes* are pairs of prime numbers p and q that have a difference of 2. Use part (a) to prove that 5 is the only prime number that takes part in two different twin prime pairs.

7 Pythagorean Theorem



Using the above diagram, prove the Pythagorean Theorem: if a and b are the lengths of the legs of a right triangle and c is the length of its hypotenuse, then $a^2 + b^2 = c^2$.
Hint: Look for right triangles in the diagram and label them with a, b, and c.

8 Inductive Charging Lemma

There are n cars on a circular track. Among all of them, they have exactly enough fuel (in total) for one car to circle the track.
Prove, using whatever method you want, that there exists at least one car that has enough fuel to reach the next car along the track.

9 Triangle Inequality

Recall the triangle inequality, which states that for real numbers x_1 and x_2,

$$|x_1 + x_2| \leq |x_1| + |x_2|.$$

Assuming the above inequality holds, use induction to prove the generalized triangle inequality:

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|.$$

10 Counterfeit Coins

(a) Suppose you have 9 gold coins that look identical, but you also know one (and only one) of them is counterfeit. The counterfeit coin weighs slightly less than the others. You also have access to a balance scale to compare the weight of two sets of coins — i.e., it can tell you whether one set of coins is heavier, lighter, or equal in weight to another (and no other information). However, your access to this scale is very limited.

Can you find the counterfeit coin using *just two weighings*? Prove your answer.

(b) Now consider a generalization of the same scenario described above. You now have 3^n coins, $n \geq 1$, only one of which is counterfeit. You wish to find the counterfeit coin with just n weighings. Can you do it? Prove your answer.