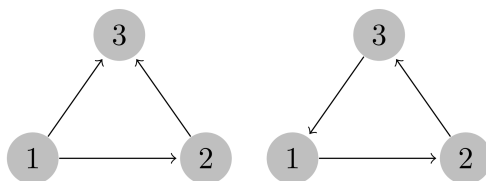Due: Friday, April 10, 2020 at 11:59 PM
Grace period until Sunday, April 12, 2020 at 11:59PM

1 Random Tournaments

A *tournament* is a directed graph in which every pair of vertices has exactly one directed edge between them—for example, here are two tournaments on the vertices $\{1,2,3\}$:

In the first tournament above, $(1,2,3)$ is a *Hamiltonian path*, since it visits all the vertices exactly once, without repeating any edges, but $(1,2,3,1)$ is not a valid *Hamiltonian cycle*, because the tournament contains the directed edge $1 \to 3$ and not $3 \to 1$. In the second tournament, $(1,2,3,1)$ is a *Hamiltonian cycle*, as are $(2,3,1,2)$ and $(3,1,2,3)$; for this problem we'll say that these are all different Hamiltonian cycles, since their start/end points are different.

Consider the following way of choosing a random tournament T on n vertices: independently for each (unordered) pair of vertices $\{i,j\} \subset \{1,...,n\}$, flip a coin and include the edge $i \to j$ in the graph if the outcome is heads, and the edge $j \to i$ if tails. What is the expected number of Hamiltonian paths in T? What is the expected number of Hamiltonian cycles?

2 Triangles in Random Graphs

Let's say we make a simple and undirected graph G on n vertices by randomly adding m edges, without replacement. In other words, we choose the first edge uniformly from all $\binom{n}{2}$ possible edges, then the second one uniformly from among the remaining $\binom{n}{2} - 1$ edges, etc. What is the expected number of triangles in G? (A triangle is a triplet of distinct vertices with all three edges present between them.)

3 Student Life

In an attempt to avoid having to do laundry often, Marcus comes up with a system. Every night, he designates one of his shirts as his dirtiest shirt. In the morning, he randomly picks one of his shirts to wear. If he picked the dirtiest one, he puts it in a dirty pile at the end of the day (a shirt

in the dirty pile is not used again until it is cleaned). When Marcus puts his last shirt into the dirty pile, he finally does his laundry, and again designates one of his shirts as his dirtiest shirt (laundry isn't perfect) before going to bed. This process then repeats.

(a) If Marcus has n shirts, what is the expected number of days that transpire between laundry events? Your answer should be a function of n involving no summations.

(b) Say he gets even lazier, and instead of organizing his shirts in his dresser every night, he throws his shirts randomly onto one of n different locations in his room (one shirt per location), designates one of his shirts as his dirtiest shirt, and one location as the dirtiest location. In the morning, if he happens to pick the dirtiest shirt, *and* the dirtiest shirt was in the dirtiest location, then he puts the shirt into the dirty pile at the end of the day and does not throw any future shirts into that location and also does not consider it as a candidate for future dirtiest locations (it is too dirty). What is the expected number of days that transpire between laundry events now? Again, your answer should be a function of n involving no summations.

4 Class Enrollment

Lydia has just started her CalCentral enrollment appointment. She needs to register for a marine science class and CS 70. There are no waitlists, and she can attempt to enroll once per day in either class or both. The CalCentral enrollment system is strange and picky, so the probability of enrolling successfully in the marine science class on each attempt is μ and the probability of enrolling successfully in CS 70 on each attempt is λ. Also, these events are independent.

(a) Suppose Lydia begins by attempting to enroll in the marine science class everyday and gets enrolled in it on day M. What is the distribution of M?

(b) Suppose she is not enrolled in the marine science class after attempting each day for the first 5 days. What is the conditional distribution of M given $M > 5$?

(c) Once she is enrolled in the marine science class, she starts attempting to enroll in CS 70 from day $M + 1$ and gets enrolled in it on day C. Find the expected number of days it takes Lydia to enroll in both the classes, i.e. $\mathbb{E}[C]$.

(d) Suppose instead of attempting one by one, Lydia decides to attempt enrolling in both the classes from day 1. Let M be the number of days it takes to enroll in the marine science class, and C be the number of days it takes to enroll in CS 70. What is the distribution of M and C now? Are they independent?

(e) Let X denote the day she gets enrolled in her first class and let Y denote the day she gets enrolled in both the classes. What is the distribution of X?

(f) What is the expected number of days it takes Lydia to enroll in both classes now, i.e. $\mathbb{E}[Y]$.

(g) What is the expected number of classes she will be enrolled in by the end of 14 days?

5 Shuttles and Taxis at Airport

In front of terminal 3 at San Francisco Airport is a pickup area where shuttles and taxis arrive according to a Poisson process. The shuttles arrive at a rate $\lambda_1 = 1/20$ (i.e. 1 shuttle per 20 minutes) and the taxis arrive at a rate $\lambda_2 = 1/10$ (i.e. 1 taxi per 10 minutes) starting at 00:00. The shuttles and the taxis arrive independently.

(a) What is the distribution of the following:

    (i) The number of taxis that arrive between times 00:00 and 00:20?

    (ii) The number of shuttles that arrive between times 00:00 and 00:20?

    (iii) The total number of pickup vehicles that arrive between times 00:00 and 00:20?

(b) What is the probability that exactly 1 shuttle and 3 taxis arrive between times 00:00 and 00:20?

(c) Given that exactly 1 pickup vehicle arrived between times 00:00 and 00:20, what is the conditional probability that this vehicle was a taxi?

(d) Suppose you reach the pickup area at 00:20. You learn that you missed 3 taxis and 1 shuttle in those 20 minutes. What is the probability that you need to wait for more than 10 mins until either a shuttle or a taxi arrives?

6 Make Your Own Question

Make your own question on this week's material and solve it.

7 Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student! We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

1. **What sources (if any) did you use as you worked through the homework?**

2. **If you worked with someone on this homework, who did you work with?** List names and student ID's. (In case of homework party, you can also just describe the group.)

3. **How did you work on this homework?** (For example, *I first worked by myself for 2 hours, but got stuck on problem 3, so I went to office hours. Then I went to homework party for a few hours, where I finished the homework.*)

4. **Roughly how many total hours did you work on this homework?**