HW 3

Due: Friday, February 14, 2020 at 11:59 PM
Grace period until Sunday, February 16, 2020 at 11:59 PM

1 A Better Stable Pairing

In this problem we examine a simple way to *merge* two different solutions to a stable matching problem. Let R, R' be two distinct stable pairings. Define the new pairing $R \wedge R'$ as follows:

> For every job j, j's partner in $R \wedge R'$ is whichever is better (according to j's preference list) of their partners in R and R'.

Also, we will say that a job/candidate *prefers* a pairing R to a pairing R' if they prefers their partner in R to their partner in R'. We will use the following example:

jobs	preferences	candidates	preferences
A	1>2>3>4	1	D>C>B>A
B	2>1>4>3	2	C>D>A>B
C	3>4>1>2	3	B>A>D>C
D	4>3>2>1	4	A>B>D>C

(a) $R = \{(A,4),(B,3),(C,1),(D,2)\}$ and $R' = \{(A,3),(B,4),(C,2),(D,1)\}$ are stable pairings for the example given above. Calculate $R \wedge R'$ and show that it is also stable.

(b) Prove that, for any pairings R, R', no job prefers R or R' to $R \wedge R'$.

(c) Prove that, for any stable pairings R, R' where j and c are partners in R but not in R', one of the following holds:

- j prefers R to R' and c prefers R' to R; or
- j prefers R' to R and c prefers R to R'.

[*Hint*: Let J and C denote the sets of jobs and candidates respectively that prefer R to R', and J' and C' the sets of jobs and candidates that prefer R' to R. Note that $|J| + |J'| = |C| + |C'|$. (Why is this?) Show that $|J| \le |C'|$ and that $|J'| \le |C|$. Deduce that $|J'| = |C|$ and $|J| = |C'|$. The claim should now follow quite easily.]

(You may assume this result in the next part even if you don't prove it here.)

(d) Prove an interesting result: for any stable pairings R, R', (i) $R \wedge R'$ is a pairing [*Hint*: use the results from (c)], and (ii) it is also stable.

2 Pairing Up

Prove that for every even $n \geq 2$, there exists an instance of the stable matching problem with n jobs and n candidates such that the instance has at least $2^{n/2}$ distinct stable matchings.

3 Well-Ordering Principle

In this question we walk you through one possible method to prove the well-ordering principle using induction. Recall that the well-ordering principle can be stated as follows:

For every non-empty subset S of the set of natural numbers $\mathbb{N}$, there is a smallest element $x \in S$; i.e.

$$\exists x : \forall y \in S : x \leq y.$$

We break this problem into cases depending on whether S is finite or not.

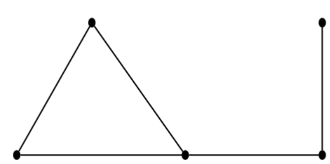
(a) Prove the well-ordering principle for finite sets S by induction on the size of the set.

(b) Prove that a set S containing only elements less than or equal to n must have size at most $n+1$, by induction on the maximal element of S.

(c) Use the previous part to prove the well-ordering principle for infinite sets S. (*Hint*: Choose an arbitrary element $x \in S$ and split S into two sets - one with all elements which are at most x and one with all elements larger than x.)

4 Degree Sequences

The *degree sequence* of a graph is the sequence of the degrees of the vertices, arranged in descending order, with repetitions as needed. For example, the degree sequence of the following graph is $(3,2,2,2,1)$.



For each of the parts below, determine if there exists a simple undirected graph G (i.e. a graph without self-loops and multiple-edges) having the given degree sequence. Justify your claim.

(a) $(3,3,2,2)$

(b) $(3,2,2,2,2,1,1)$

(c) $(6,2,2,2)$

(d) $(4,4,3,2,1)$

5 Planarity and Graph Complements

Let $G=(V,E)$ be an undirected graph. We define the complement of G as $\overline{G}=(V,\overline{E})$ where $\overline{E}=\{(i,j)|i,j\in V, i\neq j\}-E$; that is, $\overline{G}$ has the same set of vertices as G, but an edge e exists is $\overline{G}$ if and only if it does not exist in G.

(a) Suppose G has v vertices and e edges. How many edges does $\overline{G}$ have?

(b) Prove that for any graph with at least 13 vertices, G being planar implies that $\overline{G}$ is non-planar.

(c) Now consider the converse of the previous part, i.e., for any graph G with at least 13 vertices, if $\overline{G}$ is non-planar, then G is planar. Construct a counterexample to show that the converse does not hold.

Hint: Recall that if a graph contains a copy of K_5, then it is non-planar. Can this fact be used to construct a counterexample?

6 Trees and Components

(a) Bob removed a degree 3 node from an n-vertex tree. How many connected components are there in the resulting graph? Please provide an explanation.

(b) Given an n-vertex tree, Bob added 10 edges to it and then Alice removed 5 edges. If the resulting graph has 3 connected components, how many edges must be removed in order to remove all cycles from the resulting graph? Please provide an explanation.

7 Bipartite Graphs

An undirected graph is bipartite if its vertices can be partitioned into two disjoint sets L, R such that each edge connects a vertex in L to a vertex in R (so there does not exist an edge that connects two vertices in L or two vertices in R).

(a) Suppose that a graph G is bipartite, with L and R being a bipartite partition of the vertices. Prove that $\sum_{v\in L}\deg(v)=\sum_{v\in R}\deg(v)$.

(b) Suppose that a graph G is bipartite, with L and R being a bipartite partition of the vertices. Let s and t denote the average degree of vertices in L and R respectively. Prove that $s/t=|R|/|L|$.

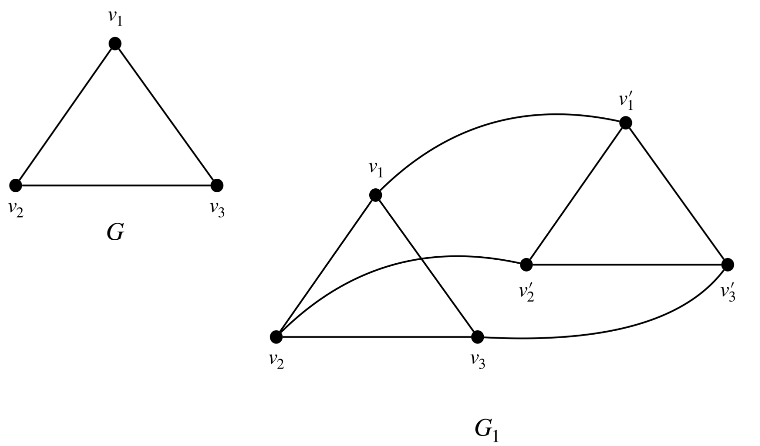
(c) The *double* of a graph G consists of two copies of G with edges joining the corresponding "mirror" vertices. More precisely, if $G=(V,E)$, where $V=\{v_1,v_2,\dots,v_n\}$ is the set of vertices and E the set of edges, then the double of the graph G is given by $G_1=(V_1,E_1)$, where

$$V_1=\{v_1,v_2,\dots,v_n,v_1',v_2',\dots,v_n'\},$$

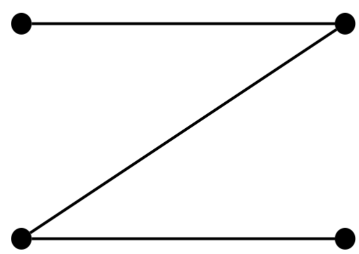
and

$$E_1=E\cup\{(v_i',v_j')|(v_i,v_j)\in E\}\cup\{(v_i,v_i'),\forall i\}.$$

Here is an example,



Draw the double of the following graph:



(d) Now suppose that G_1 is a bipartite graph, G_2 is the double of G_1, G_3 is the double of G_2, and so on. (Each G_{i+1} has twice as many vertices as G_i). Show that $\forall n \geq 1$, G_n is bipartite.

Hint: Use induction on n.

8 The Last Digit

In each case show your work and justify your answers.

(a) If $9k+5$ and $2k+1$ have the same last digit for some natural number k, find the last digit of k.

(b) If $S = \sum_{i=1}^{19} i!$, then find the last digit of S^2.

(c) Denote the last digit of a natural number a by b. Show that the last digit of a^n is the same as the last digit of b^n where is $n \geq 1$ is a natural number.

(d) Inspired by part (c), show that the last digit of a^{4k+1} for all natural numbers k is the same as the last digit of a. [Euler's Theorem is not allowed.]

9 Can You Invert It?

State whether each of the following functions f have a well-defined inverse or not. Recall that a function has an inverse if, for every element in the codomain, there exists exactly one element in the domain that maps to it; that is, the function is onto (surjective) and one-to-one (injective). If it does, provide the inverse function f^{-1}. If it does not, explain if it violates the onto condition, one-on-one condition, or both, and justify your response.

(a) $f : \mathbb{R} \to \mathbb{R},\ f(x) = 3x - 12$

(b) $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2 + 1$

(c) $f : \mathbb{R} \to \mathbb{R},\ f(x) = 3x^3 - 12x$

(d) $f : \mathbb{R}^+ \to \mathbb{R}^+,\ f(x) = \sqrt{x+1}$, where $\mathbb{R}^+$ denotes the positive reals

(e) $f : \mathbb{R}^+ \to \mathbb{N},\ f(x) = \lfloor x \rfloor$

(f) $f : \mathbb{R}^3 \to \mathbb{R}^3,\ f(x) = Ax$, where $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(g) $f : \mathbb{R}^3 \to \mathbb{R}^3,\ f(x) = Ax$, where $A = \begin{bmatrix} 5 & 2 & 4 \\ 1 & 0 & 2 \\ 3 & 2 & 0 \end{bmatrix}$

10 Make Your Own Question

Make your own question on this week's material and solve it.

11 Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student! We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

1. **What sources (if any) did you use as you worked through the homework?**
2. **If you worked with someone on this homework, who did you work with?** List names and student ID's. (In case of homework party, you can also just describe the group.)
3. **How did you work on this homework?** (For example, *I first worked by myself for 2 hours, but got stuck on problem 3, so I went to office hours. Then I went to homework party for a few hours, where I finished the homework.*)
4. **Roughly how many total hours did you work on this homework?**