CS 70 Discrete Mathematics and Probability Theory
Spring 2020 Course Notes **HW 7**

Due: Friday, March 13, 2020 at 11:59 PM
Grace period until Sunday, March 15, 2020 at 11:59 PM

1 Stirling's Approximation

In this question, suppose $n \in \mathbb{Z}^+$, we want to find approximations for $n!$. To plot the given functions, you can use an online tool (*e.g.*, go to http://www.wolframalpha.com/ and type "plot $\ln x$").

1. Plot the function $f(x) = \ln x$.
2. For the following three questions, please note that $\ln x$ is strictly increasing and concave-$\cap$ because, when $x > 0$, its first and second derivatives are positive and negative, respectively. Concavity means that all line segments connecting two points on the function are below the function.
   Suppose $n \in \mathbb{Z}^+$, use the plot to explain why
   $$\ln 1 + \ln 2 + \ldots + \ln n \geq \int_1^n \ln x dx \tag{1}$$
3. Suppose $n \in \mathbb{Z}^+$, use the plot to explain why
   $$\ln 1 + \ln 2 + \ldots + \ln n < \int_1^{n+1} \ln x dx \tag{2}$$
4. Suppose $a \in \mathbb{Z}^+$, use the plot to explain why
   $$\left(\frac{\ln a + \ln(a+1)}{2}\right) < \int_a^{a+1} \ln x dx \tag{3}$$
5. Use Equation ((1)) to prove $n! \geq e\left(\frac{n}{e}\right)^n$.
6. Use Equation ((2)) to prove $n! \leq en\left(\frac{n}{e}\right)^n$ (Hint: If in this part you find yourself wishing you had $n-1!$ on the left-hand-side, try to prove an upper bound on $n-1!$ and use that to help you)
7. Use Equation ((3)) to prove $n! \leq e\sqrt{n}\left(\frac{n}{e}\right)^n$, which is a tighter upper bound.
8. The Stirling's approximation is usually written as $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ or a simpler version $n! \approx \left(\frac{n}{e}\right)^n$. Plot the function $f(n) = \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{n!}$. What do you observe?

9. Suppose $m = \frac{k}{n}$, use m, n and apply the simpler version of the Stirling's approximation to rewrite $\binom{n}{k}$.

10. Now, suppose $m_1 = \frac{k_1}{n} = 0.25$, $m_2 = \frac{k_2}{n} = 0.5$, and $m_3 = \frac{k_3}{n} = 0.75$, plot $\ln(\binom{n}{k_1})$, $\ln(\binom{n}{k_2})$, and $\ln(\binom{n}{k_3})$ as functions of n on a plot with linear-scaled axes. What do you observe?

2 Double Injections

The zero-th law of counting says that if a bijection (one-to-one and onto mapping) exists between two sets A and B, the two sets have the same size. In some cases, rather than coming up with a single bijection, it is easier to come up with two different one-to-one mappings. One of them going from A to B, and the other going from B to A.

The spirit of the pigeon-hole principle intuitively tells us that if we can find an assignment of pigeons to holes such that no two pigeons are in the same hole and every pigeon has a hole, as well as a possibly distinct assignment of holes to pigeons so that no two holes are assigned to the same pigeon and every hole has a pigeon, then there must be the same number of holes as pigeons. But this intuitive fact actually needs a proof. (Especially if we want to consider infinite sets.) This fact is called the Cantor-Schröder-Bernstein theorem, and this problem walks you through the proof of it.

Assume that there exist injective (one-to-one) functions $f : A \to B$ and $g : B \to A$.

The key here is to somehow construct a bijection h from these two maps. Our goal is to reuse the functions f and g as much as possible, but we have to confront the key question of when do we use f and when do we use g.

To navigate this key question, we define the *chain* of an element $a \in A$ as the following sequence:

$$\cdots \to f^{-1}(g^{-1}(a)) \to g^{-1}(a) \to a \to f(a) \to g(f(a)) \to \cdots$$

And similarly, the chain of an element $b \in B$ as:

$$\cdots \to g^{-1}(f^{-1}(b)) \to f^{-1}(b) \to b \to g(b) \to f(g(b)) \to \cdots$$

We will use the notation $C(a)$ to denote the chain of a. (And similarly for b.) Note that $C(a)$ will always extend infinitely to the right of a (i.e. the sequence $a, f(a), g(f(a)), \cdots$ does not terminate), because we can always apply f and g to elements in A and B, respectively. However, the same is not true when extending to the left of a, because the inverse mappings g^{-1} and f^{-1} may not exist for certain elements in A or B. After all, neither of the mappings f and g are promised to be onto mappings. They are only guaranteed to be one-to-one.

It turns out that there are four types of chains:

1. **Cyclic Chains** form a loop. For example, the sequence $a_1 \to b_1 \to a_2 \to b_2 \to a_1$ is cyclic.

2. **Doubly Infinite Chains** extend infinitely in the leftwards direction.

3. **A-Stoppers** end (on the left) in A. That is, an $f^{-1}(\cdot)$ lands us in an element of A that isn't in the range of g.

4. **B-Stoppers** end (on the left) in B. That is, a $g^{-1}(\cdot)$ lands us in an element of B that isn't in the range of f.

(a) Our definition of a cyclic chain above is quite restrictive — it requires the entire chain to form a single loop. Indeed, if there is ever a cycle in a chain, the entire chain is always a single loop. **Argue why this is always the case.**

(HINT: For example, the sequence $a_1 \to b_1 \to a_2 \to b_2 \to a_3 \to b_1$, where all elements are distinct, is not a valid chain. Why?)

(b) Two chains are distinct if one contains any element that the other one does not. They are the same chain if they contain all the same elements. **Prove that every element of A and B is part of exactly one chain.** Note that many elements can share the same chain (e.g. a and $f(a)$ are always in the same chain).

(HINT: You need to show that if two chains intersect at all, they must be the same chain.)

(c) We will now try to demonstrate that a bijection $h : A \to B$ exists. Since every element of A and B is part of exactly one chain, we will try to demonstrate that h will be a bijection for chains of a particular type. That is, to prove that h is a bijection for chains of type T, we will demonstrate that, for all chains C of type T, the function h is a bijection from C_A (the subset of A in the chain C) to C_B (the subset of B in the chain C).

Your friend suggests using f as a candidate bijection (i.e. $h(x) = f(x)$). **For which types of chains will f be a bijection? For which types will f not be a bijection? Justify your answer, for each case: prove that f is a bijection, or argue why f need not be a bijection.**

(d) In the previous part, you will have noticed that f is not necessarily a bijection when a belongs to a B-stopper. **Demonstrate that for all chains C that are B-Stoppers, g^{-1} exists and is a bijection from C_A to C_B.**

(e) Using what you've learned in the previous parts, you sugggest the following function as a candidate bijection:

$$h(x) = \begin{cases} g^{-1}(a) & \text{,if } a \text{ is part of a B-Stopper} \\ f(a) & \text{,otherwise} \end{cases}$$

Demonstrate that h is now a bijection between A and B.

(*Hint: Use your analysis from the previous parts.*)

3 Counting, Counting, and More Counting

The only way to learn counting is to practice, practice, practice, so here is your chance to do so. For this problem, you do not need to show work that justifies your answers. We encourage you to leave your answer as an expression (rather than trying to evaluate it to get a specific number).

(a) How many ways are there to arrange n 1s and k 0s into a sequence?

(b) A bridge hand is obtained by selecting 13 cards from a standard 52-card deck. The order of the cards in a bridge hand is irrelevant.
How many different 13-card bridge hands are there? How many different 13-card bridge hands are there that contain no aces? How many different 13-card bridge hands are there that contain all four aces? How many different 13-card bridge hands are there that contain exactly 6 spades?

(c) Two identical decks of 52 cards are mixed together, yielding a stack of 104 cards. How many different ways are there to order this stack of 104 cards?

(d) How many 99-bit strings are there that contain more ones than zeros?

(e) An anagram of FLORIDA is any re-ordering of the letters of FLORIDA, i.e., any string made up of the letters F, L, O, R, I, D, and A, in any order. The anagram does not have to be an English word.
How many different anagrams of FLORIDA are there? How many different anagrams of ALASKA are there? How many different anagrams of ALABAMA are there? How many different anagrams of MONTANA are there?

(f) How many different anagrams of ABCDEF are there if: (1) C is the left neighbor of E; (2) C is on the left of E (and not necessarily E's neighbor)

(g) We have 9 balls, numbered 1 through 9, and 27 bins. How many different ways are there to distribute these 9 balls among the 27 bins? Assume the bins are distinguishable (e.g., numbered 1 through 27).

(h) We throw 9 identical balls into 7 bins. How many different ways are there to distribute these 9 balls among the 7 bins such that no bin is empty? Assume the bins are distinguishable (e.g., numbered 1 through 7).

(i) How many different ways are there to throw 9 identical balls into 27 bins? Assume the bins are distinguishable (e.g., numbered 1 through 27).

(j) There are exactly 20 students currently enrolled in a class. How many different ways are there to pair up the 20 students, so that each student is paired with one other student?

(k) How many solutions does $x_0 + x_1 + \cdots + x_k = n$ have, if each x must be a non-negative integer?

(l) How many solutions does $x_0 + x_1 = n$ have, if each x must be a *strictly positive* integer?

(m) How many solutions does $x_0 + x_1 + \cdots + x_k = n$ have, if each x must be a *strictly positive* integer?

4 Shipping Crates

A widget factory has four loading docks for storing crates of ready-to-ship widgets. Suppose the factory produces 8 indistinguishable crates of widgets and sends each crate to one of the four loading docks.

(a) How many ways are there to distribute the crates among the loading docks?

(b) Now, assume that any time a loading dock contains at least 5 crates, a truck picks up 5 crates from that dock and ships them away. (e.g., if 6 crates are sent to a loading dock, the truck removes 5, leaving 1 leftover crate still in the dock). We will now consider two configurations to be identical if, for every loading dock, the two confirgurations have the same number of leftover crates in that dock. How would your answer in the previous part compare to the number of outcomes given the new setup? Justify your answer.

(c) We will now attempt to count the number of configurations of crates. First, we look at the case where crates are removed from the dock. How many ways are there to distribute the crates such that some crate gets removed from the dock?

(d) How many ways are there to distribute the crates such that no crates are removed from the dock; i.e. no dock receives at least 5 crates?

(e) Putting it together now, what are the total number of possible configurations for crates in the modified scenario? *Hint:* Observe that, regardless of which dock receives the 5 crates, we end up in the same situation.

After all the shipping has been done, how many possible configurations of leftover crates in loading docks are there?

5 Binomial Theorem

Imagine you are throwing n numbered balls into bins. There are x red bins and y blue bins.

Use the above scenario in a combinatorial argument to prove the binomial theorem, which states the following:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

6 Charm School Applications

(a) n males and n females apply to the Elegant Etiquette Charm School (EECS) within UC Berkeley. The EECS department only has n seats available. In how many ways can it admit students? Use the above story for a combinatorial argument to prove the following identity:

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2$$

(b) Among the n admitted students, there is at least one male and at least one female. On the first day, the admitted students decide to carpool to school. The boy(s) get in one car, and the girl(s) get in another car. Use the above story for a combinatorial argument to prove the following identity:

$$\sum_{k=1}^{n-1} k \cdot (n-k) \cdot \binom{n}{k}^2 = n^2 \cdot \binom{2n-2}{n-2}$$

(Hint: Each car has a driver...)

7 Fermat's Wristband

Let p be a prime number and let k be a positive integer. We have beads of k different colors, where any two beads of the same color are indistinguishable.

(a) We place p beads onto a string. How many different ways are there construct such a sequence of p beads with up to k different colors?

(b) How many sequences of p beads on the string are there that use at least two colors?

(c) Now we tie the two ends of the string together, forming a wristband. Two wristbands are equivalent if the sequence of colors on one can be obtained by rotating the beads on the other. (For instance, if we have $k = 3$ colors, red (R), green (G), and blue (B), then the length $p = 5$ necklaces RGGBG, GGBGR, GBGRG, BGRGG, and GRGGB are all equivalent, because these are all rotated versions of each other.)

   How many non-equivalent wristbands are there now? Again, the p beads must not all have the same color. (Your answer should be a simple function of k and p.)

   [*Hint*: Think about the fact that rotating all the beads on the wristband to another position produces an identical wristband.]

(d) Use your answer to part (c) to prove Fermat's little theorem.

8 Make Your Own Question

Make your own question on this week's material and solve it.

9 Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student! We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

1. **What sources (if any) did you use as you worked through the homework?**
2. **If you worked with someone on this homework, who did you work with?** List names and student ID's. (In case of homework party, you can also just describe the group.)

3. **How did you work on this homework?** (For example, *I first worked by myself for 2 hours, but got stuck on problem 3, so I went to office hours. Then I went to homework party for a few hours, where I finished the homework.*)

4. **Roughly how many total hours did you work on this homework?**