EECS 70 Discrete Mathematics and Probability Theory
Fall 2014 Anant Sahai

Midterm 1

Exam location: 10 Evans, Last name starting with A-B or R-T

PRINT your student ID: ____

PRINT AND SIGN your name: ________________, ________________ ____________________
(last) (first) (signature)

PRINT your Unix account login: cs70-_____________

PRINT your discussion section and GSI (the one you attend): ____

Name of the person to your left: ____

Name of the person to your right: ____

Name of the person in front of you: ____

Name of the person behind you: ____

Section 0: Pre-exam questions (3 points)

1. **What other courses are you taking this term? (1 pt)**

2. **What activity do you really enjoy? Describe how it makes you feel. (2 pts)**

Do not turn this page until the proctor tells you to do so. You can work on Section 0 above before time starts.

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Section 1: Straightforward questions (24 points)

Unless told otherwise, you must show work to get credit. You get one drop: do 4 out of the following 5 questions. (We will grade all 5 and keep only the best 4 scores) However, there will be essentially no partial credit given in this section. Students who get all 5 questions correct will receive some bonus points.

3. XOR

The truth table of XOR is as follows.

A	B	A XOR B
F	F	F
F	T	T
T	F	T
T	T	F

(a) **Express XOR using only** $(\wedge, \vee, \neg)$ **and parentheses.**

(b) **Does** $(A$ **XOR** $B)$ **imply** $(A \vee B)$**? Explain briefly.**

(c) **Does** $(A \vee B)$ **imply** $(A$ **XOR** $B)$**? Explain briefly.**

4. Stable Marriage

Consider the set of men $M = \{1, 2, 3\}$ and the set of women $W = \{A, B, C\}$ with the following preferences.

Men	Women		
1	A	B	C
2	B	A	C
3	A	B	C

Women	Men		
A	2	1	3
B	1	2	3
C	1	2	3

Run the male propose-and-reject algorithm on this example. How many days does it take and what is the resulting pairing? (Show your work)

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[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

5. Prove it

Suppose x, y are integers. **Prove that if 5 does not divide xy, then 5 does not divide x and 5 does not divide y.**

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6. Inequality

Prove *by induction* on n that if n is a natural number and $x > 0$, then $(1+x)^n \geq 1+nx$.

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

7. Stable Marriage

Below are the observed proposals from the traditional male propose-and-reject algorithm.

Day	Women	Men
1	A	-
	B	1, 2
	C	3
	D	4
2	A	-
	B	1
	C	2, 3
	D	4
3	A	-
	B	1
	C	2
	D	3, 4
4	A	-
	B	1, 4
	C	2
	D	3

Day	Women	Men
5	A	-
	B	4
	C	2
	D	1, 3
6	A	-
	B	3, 4
	C	2
	D	1
7	A	-
	B	3
	C	2, 4
	D	1
8	A	2
	B	3
	C	4
	D	1

After the algorithm terminates, only B is married to the man she likes the most. Also, it is known that A has the same preferences as B and every man likes C better than A.

Reconstruct the complete preference lists of men and women given the information above. (You do not have to show work.)

Men	Preferences
1	> > >
2	> > >
3	> > >
4	> > >

Women	Preferences
A	> > >
B	> > >
C	> > >
D	> > >

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Section 2: True/False (30 points)

For the questions in this section, determine whether the statement is true or false. If true, prove the statement is true. If false, provide a counterexample demonstrating that it is false.

You get one drop: do 2 out of the following 3 questions. (We will grade all three questions and keep only the best two scores.) Students who get all three questions perfectly correct will receive some bonus points.

8. Sets (15 points)

If $n > 0$ is a positive integer, and S is a set of distinct positive integers, all of which are less than or equal to n, then S has at most n elements.

Mark one: TRUE or FALSE.

9. Does "No" Matter? (15 points)

Consider an alternative to the Propose and Reject algorithm (with no rejections), where women take turns choosing the best available husband from the remaining unchosen men. On day 1, the oldest woman chooses her most preferred man, and marries him. On day k, the k-th eldest woman chooses her most preferred choice from the remaining unmarried men, and marries him. **No matter what the preferences are, this process always results in a stable matching.**

Mark one: TRUE or FALSE.

10. ***n*** **Matchings (15 points)**

For all positive n, it is always possible to construct a set of preferences for n women and n men such that at least n distinct stable matchings are possible.

Mark one: TRUE or FALSE.

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Section 3: Free-form Problems (65 points)

11. Bieber Fever (35 points)

In this world, there are only two kinds of people: people who love Justin Bieber, and people who hate him. We are searching for a stable matching for everyone. The situation is as follows:

- For some $n \geq 5$, there are n men, n women, and one Justin Bieber[1].
- Men can be matched with women; or anyone can be matched with Justin Bieber.
- Everyone is either a Hater or a Belieber. Haters want to be matched with anyone but Justin Bieber. Beliebers really want to be matched with Justin Bieber but don't mind being matched with other people.
- Men and women still have preference lists, as usual, but if they are a Belieber, Justin Bieber is always in the first position. If they are a hater, Justin Bieber is always in the last position.
- Justin Bieber desires to have 10 individuals matched with him (to party forever). As Justin Bieber is a kind person and wishes to be inclusive, he wishes to have exactly 5 women and 5 men in his elite club.
- Justin Bieber also has a preference list containing all $2n$ men and women.

A stable matching is defined as follows:

- Justin Bieber has 10 partners, of which 5 are men and 5 are women.
- All men and women not matched up with Justin Bieber are married to someone of the opposite gender.
- No rogue couples exist; i.e., there is no man M and woman W such that M prefers W to his current wife, and W prefers M to her current husband.
- No Hater is matched with Justin Bieber.
- There is no man who (1) is not matched with Justin Bieber; and (2) who is prefered by Justin Bieber over one of his current male partners; and (3) who prefers Justin Bieber over his wife. And similarly for women vis-a-vis Justin Bieber relative to their husbands and Justin Bieber's female partners.

(a) (5 points) **Show that there does not necessarily exist a stable matching.**

[1] For the purposes of this problem, Justin Bieber is neither male nor female.

(b) (10 points) **Provide an "if-and-only-if" condition for whether a stable matching exists.** (*No need to prove anything in this part. That comes in later parts of this question.*)

(c) (5 points) **Is Justin Bieber guaranteed to always get his Bieber-optimal group if a stable matching exists?** (Bieber-optimal means that he gets the best possible group that could be matched to him in any stable matching.)

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(d) (10 points) **Give an algorithm which finds a stable matching if the condition you gave in (b) holds. Argue why this algorithm works.**

(e) (5 points) **Prove that a stable matching cannot exist if the condition you gave in (b) does not hold.**

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12. Tournament (30 points)

A Round Robin Tournament (RRT) between n people $A_1, \ldots, A_n$ is a tournament where every person plays every other person exactly once, and games never end in a tie. For example, an RRT between 3 people could result in

$$A_1 \to A_2$$
$$A_1 \to A_3$$
$$A_2 \leftarrow A_3$$

which is to say A_1 beat A_2 and A_3, and A_3 beat A_2. Suppose you only know how many wins everyone had. That is, person A_i only tells you W_i, their total number of wins.

(a) (5 points) **Show that such a tournament can have at most 1 person with 0 wins.**

(b) (5 points) **Show that such a tournament can have at most 1 person with $n-1$ wins.**

(c) (10 points) **Prove that if each of the W_i's is unique, you can tell exactly who beat who.**

(HINT: It might be helpful to work out examples for $n = 2, 3, 4$ to help you see what is going on.)

(d) (10 points) **Prove that if you can tell exactly who beat who, then each of the W_i's is unique.**

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13. (optional) Matchmaking Cruise (20 points)

Mr. and Mrs. Matchmaker are sponsoring a series of matchmaking cruises for single women. There are n women and n men with preference lists for each other. Assume n is even. The Matchmakers guarantee a spot on the ship for all n women, but can only fit $n/2$ men at a time. Since space is limited, the Matchmakers decide to let all n women aboard but the men are divided into two groups of $n/2$ men, Group A and Group B.

For the first "week," men from Group A are allowed to come aboard and start proposing to the n women through the male Propose and Reject algorithm, until no man receives any more rejections. (Just assume that a "week" is long enough for this to happen.) For the second week, Group A leaves and Group B starts proposing until no rejections are received. On the third week, Group A returns and this process continues to repeat until no man from either Group A or Group B is rejected.

During this process, if a woman still had an active proposal in hand from man a from Group A at the end of a particular week, then the next week, she will reject man b from Group B if she prefers a over b. On the other hand, if she prefers b over a, she will say "maybe" to b and reject a when he returns to the cruise the next week and re-proposes to her.

State and prove an Improvement Lemma for this scenario.

(This Improvement Lemma should be sufficiently powerful to be able to be used to eventually get a proof that the Matchmakers will end up with a stable pairing using their cruises.)

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[Doodle page! Draw us something if you want or give us suggestions or complaints. You can also use this page to report anything suspicious that you might have noticed.]